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Stability Analysis for Systems with Impulse Effects

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In the present paper the authors establish new conditions for the uniform stability and the uniform asymptotic stability of equilibria of systems with impulsive effects described by systems of nonlinear, time-varying ordinary differential equations. For the case when the corresponding systems without impulsive effects admit unstable properties, the above results are used to establish conditions under which the uniform stability even uniform asymptotic stability of equilibria of systems with impulsive effects can be caused by impulsive perturbations .

KEY WORDS: hybrid dynamical systems; lyapunov stability; systems with impulses.

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1. INTRODUCTION

There are numerous examples of evolutionary systems which at certain instants in time are subjected to rapid changes. In the simulations of such processes it is frequently convenient and valid to neglect the durations of the rapid changes and to assume that the changes can be represented by state jumps. Examples of such systems arise in mechanics (e.g., the behavior of a bouncing ball, the behavior of a buffer subjected to shock effects, the behavior of clock mechanisms, the change of velocity of a rocket at the time of separation of a stage, and so forth), in radio engineering and communication systems (where the generation of impulses of various forms is common), in biological systems (where, e.g., sudden population changes due to external effects occur frequently), in control theory (e.g., impulse control, robotics, etc.), and the like. Perhaps, the greatest current interest in such systems arises in the area of impact mechanics (see, e.g., Bainov and Simeonov, 1989) and in the study of hybrid dynamical systems (see, e.g., Branicky, 1995; Ye *et al.*, 1995a,b, 1998). For additional specific examples, refer to (Benzaid

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and Sznaier, 1994; Brogliato, 1996; Liu, 1991; Michel and Wang, 1995) and (Ye *et al.*, 1998).

Appropriate mathematical models for processes of the type described above are so-called *systems with impulsive effects*. Although analytical work for such systems abounds in the literature (which is mostly based on energy arguments and variational methods), relatively few Lyapunov stability results for such systems have been reported (see, e.g., Bainov and Simeonov, 1989; Gopalsamy and Zhang, 1989; Liu, 1991 and Ye *et al.*, 1995b; Ye *et al.*, 1998a,b). In these results, systems with impulse effects are frequently described by measure differential equations (Pandit and Deo, 1982).

In the present paper we establish new stability results (comparison to those existing results (Bainov and Simeonov, 1989; Ye *et al.*, 1998a,b) for a large class of systems with impulse effects described by systems of nonlinear, time-varying, ordinary differential equations. Our results show that certain impulsive perturbations may make a unstable system uniformly stable even uniformly asymptotically stable.

2. NOTATION

Let (X, d) be a metric space where X denotes the underlying set and d denotes the metric.

Definition 2.1. (Motion). Let $A \subset X$ and let $T \subset R^+ = [0, \infty)$. For any fixed $a \in A$, $t_0 \in T$, a mapping $p(\cdot, a, t_0) : T_{a,t_0} \to X$ is called a *motion* if $p(t_0, a, t_0) = a$ where $T_{a,t_0} = [t_0, t_1) \cap T$, $t_1 > t_0$ and t_1 is finite or infinite.

Definition 2.2. (Dynamical System). Let S be a family of motions, i.e.,

$$S \subset \{p(\cdot, a, t_0) \in \Lambda : p(t_0, a, t_0) = a\}$$

where

$$\Lambda = \cup_{(a,t_0)\in A\times T}\{T_{a,t_0} \times \{a\} \times \{t_0\} \to X\}$$

and $T_{a,t_0} \times \{a\} \times \{t_0\} \to X$ denotes a mapping from $T_{a,t_0} \times \{a\} \times \{t_0\}$ into X. The four-tuple $\{T, X, A, S\}$ is called a *dynamical system*.

To characterize the qualitative behavior of dynamical systems, we will utilize the concepts given below (refer, e.g., to Michel and Wang, 1995).

Definition 2.3. (Invariant Set). Let $\{T, X, A, S\}$ be a dynamical system. A set $M \subset A$ is said to be *invariant with respect to system S* if $a \in M$ implies that

$p(t, a, t_0) \in M$ for all $t \in T_{a,t_0}$, all $t_0 \in T$ and all $p(\cdot, a, t_0) \in S$. In particular, when $M = \{x_0\}$ for $x_0 \in A$, then x_0 is said to be an *equilibrium*.

Definition 2.4. (Lyapunov Stability). Let $\{T, X, A, S\}$ be a dynamical system and let $M \subset A$ be an invariant set of S. We say that (S, M) is *stable* if for every $\varepsilon > 0$ and $t_0 \in T$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $d(p(t, a, t_0), M) < \varepsilon$ for all $t \in T_{a,t_0}$ and for all $p(\cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$. We say that (S, M) is *uniformly stable* if $\delta = \delta(\varepsilon)$. Furthermore, if (S, M) is stable and if for any $t_0 \in T$ there exists an $\eta = \eta(t_0) > 0$ such that $\lim_{t\to\infty} d(p(t, a, t_0), M) = 0$ for all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \eta$, then (S, M) is called *asymptotically stable*. We call (S, M) *uniformly asymptotically stable* if (S, M) is uniformly stable and if there exists a $\delta > 0$ and for every $\varepsilon > 0$ there exists a $\tau = \tau(\varepsilon) > 0$ such that $d(p(t, a, t_0), M) < \varepsilon$ for all $t \in \{t \in T_{a,t_0} : d(t, t_0) \geq \tau\}$, and all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \delta$.

3. STABILITY ANALYSIS FOR SYSTEMS WITH IMPULSE EFFECTS

The present section consists of two parts. In the first of these, we present the description of the class of systems with impulse effects considered and we summarize *existing stability results* (Bainov and Simeonov, 1989; Ye *et al.*, 1998a) for this class of systems. In the second part we establish new stability results. These results show that uniform stability even uniform asymptotic stability of systems with impulsive perturbations may be caused by certain impulsive perturbations though the corresponding systems without impulsive effects admit unstable properties. Thus, we think that impulsive perturbations may give an efficient method to deal with some plants which cannot endure frequent disturbance (i.e., control input).

3.1. Systems with Impulse Effects

In the present paper we are concerned with finite-dimensional dynamical systems determined by ordinary differential equations with impulse effects. Let $X = R^n$ (R^n is the real n -space) and let d be the metric determined by the Euclidean vector norm $||\cdot||$, $||x|| = (x^T x)^{1/2}$, where $x \in R^n$. For $A \in R^{n\times n}$, let $||A||$ denote the norm of A induced by the Euclidean vector norm, i.e., $||A|| = [\lambda_{\max}(A^T A)]^{1/2}$.

The class of systems with impulse effects under investigation can be described by equations of the form

$$\begin{cases} \frac{dx}{dt} = f(x, t), & t \neq \tau_k \\ \Delta x = I_k(x), & t = \tau_k \end{cases} \tag{1}$$

where $x \in X = R^n$ denotes the state and $f \in C[R^n \times R, R^n]$ satisfies a Lipschitz condition with respect to x which guarantees the existence and uniqueness of solutions of systems (1) for given initial conditions [1]. ($C[U, W]$ denotes the set of all continuous functions from set U to set W, and $C^k[U, W]$ denotes the set of all functions from U to W which have continuous derivatives up to order k). The set $E = \{\tau_1, \tau_2, \ldots : \tau_1 < \tau_2 < \cdots\} \subset R^+$ is an unbounded, closed, discrete subset of R^+ which denotes the set of times when jumps occur, and $I_k : R^n \to R^n$ denotes the incremental change of the state at the time τ_k. It should be pointed out that in general E depends on a specific motion and that for different motions the corresponding sets E are in general different. The function $\phi : [t_0, \infty) \to R^n$ is said to be a *solution* of the system with impulse effects (1) if: 1) $\phi(t)$ is left continuous on $[t_0, \infty)$ for some $t_0 \geq 0$; 2) $\phi(t)$ is differentiable and $(d\phi/dt)(t) = f(\phi(t), t)$ everywhere on (t_0, ∞) except on an unbounded closed discrete subset $E = \{\tau_1, \tau_2, \ldots : \tau_1 < \tau_2 < \cdots\} \subset R^+$; and 3) for any $t = \tau_k \in E$, $\phi(t^+) = \lim_{s \to t, s > t} \phi(s) = \phi(t) + I_k(\phi(t))$. We will use this notation throughout this paper, i.e., for any $g : U \subset R \to R^n$, the right limit of g at $t \in U$ is denoted by $g(t^+)$, i.e., $g(t^+) = \lim_{s \to t, s > t} g(s)$.

If for system (1), we assume further that $f(0, t) = 0$ for all $t \in R^+$, and $I_k(0) = 0$ for all $k \in \mathbf{N} = \{1, 2, \ldots\}$, then it is clear that $x = 0$ is an equilibrium. For this equilibrium, the following results have been established in [1, Th. 13.1 and 13.2] and [11, Th. 3.1], respectively.

Proposition 3.1. *(Bainov and Simeonov, 1989) Assume that for system (1) satisfying* $f(0, t) = 0$ *and* $I_k(0) = 0$ *for all* $t \in R^+$ *and* $k \in \mathbf{N}$*, there exists a* $V : X \times R^+ \to R^+$ *and* $\phi_1, \phi_2 \in \boldsymbol{K}$ *such that*

$$\phi_1(||x||) \leq V(x, t) \leq \phi_2(||x||) \tag{2}$$

for all $(x, t) \in X \times R^+$. *[A function* $\phi \in C([0, r_1], R^+)$ *(respectively,* $\phi \in C(R^+, R^+)$ *) belongs to class* $\boldsymbol{K}$ *(i.e.,* $\phi \in \boldsymbol{K}$ *), if* $\phi(0) = 0$ *and if* ϕ *is strictly increasing on* $[0, r_1]$ *(respectively, on* R^+*).]*

1) If for any solution $x(t)$ *of (1) which is defined on* $[t_0, \infty)$*, it is true that* $V(x(t), t)$ *is left continuous on* $[t_0, \infty)$*, and is differentiable everywhere on* (t_0, ∞) *except on the set* $E = \{\tau_1, \tau_2, \ldots\}$*, where* E *is the set of the times when jumps occur for* $x(t)$*, and if it is also true that* $dV(x(t), t)/dt \leq 0$ *for* $t \neq \tau_k$*, and*

$$V(x(t^+), t^+) \leq V(x(t), t) \quad \text{for} \quad t = \tau_k \tag{3}$$

for all $\tau_k \in E$*, then the equilibrium* $x = 0$ *of system (1) is uniformly stable.*

2) If in addition, we assume that there exists a $\phi_3 \in \boldsymbol{K}$ *such that*

$$\frac{dV(x(t), t)}{dt} \leq -\phi_3(||x(t)||), \quad t \neq \tau_k \tag{4}$$

then the equilibrium $x = 0$ *of system (1) is uniformly asymptotically stable.*

Proposition 3.2. *(Ye et al., 1998a) Assume that for system (1)* $f(0,t) = 0$ *and* $I_k(0) = 0$ *for all* $t \in R^+$ *and* $k \in \boldsymbol{N}$*, that there exists an* $h \in C(R^+, R^+)$ *such that* $h(0) = 0$*, a* $V : X \times R^+ \to R^+$*, and* $\phi_1, \phi_2 \in \boldsymbol{K}$ *such that (2) is satisfied.*

1) Assume that for any solution $x(t)$ *of (1) which is defined on* $[t_0, \infty)$*,* $V(x(t), t)$ *is left continuous on* $[t_0, \infty)$*, and is differentiable everywhere on* (t_0, ∞) *except on the set* $E = \{\tau_1, \tau_2, \ldots\}$ *where* E *is the set of times when jumps occur for* $x(t)$*, and that* $V(x(\tau_n^+), \tau_n^+)$ *is nonincreasing for* $n = 0, 1, \ldots$ *where* $\tau_0 = t_0$*,* $I_0 = 0$*. Furthermore, assume that*

$$V(x(t), t) \le h(V(x(\tau_n^+), \tau_n^+)) \tag{5}$$

is true for all $t \in (\tau_n, \tau_{n+1}]$ *and* $n \in \boldsymbol{N}$*. Then the equilibrium* $x = 0$ *of system (1) is uniformly stable.*
2) If in addition to 1) we assume that there exists a $\phi_3 \in \boldsymbol{K}$ *such that*

$$DV(x(\tau_n^+), \tau_n^+) \le -\phi_3(||x(\tau_n^+)||) \tag{6}$$

is true for all $n \in \boldsymbol{N}$*, where*

$$DV(x(\tau_n^+), \tau_n^+) = \frac{1}{\tau_{n+1} - \tau_n}[V(x(\tau_{n+1}^+), \tau_{n+1}^+) - V(x(\tau_n^+), \tau_n^+)], \tag{7}$$

then the equilibrium $x = 0$ *of system (1) is uniformly asymptotically stable.*

The above proposition 3.1 provides a sufficient condition for the (asymptotic) stability of the equilibrium $x = 0$ of system (1). In Ye *et al.* (1998a), the authors pointed out that one critical assumption in proposition 3.1 is that the impulse effects occur at fixed instants of time, i.e., in (1) the set $E = \{\tau_1, \tau_2, \ldots\}$ is independent of the different solutions. This assumption may be unrealistic, since in applications it is often the case that the impulse effects occur when a given motion reaches some threshold condition. While in proposition 3.2, the impulse effects are considered which may occur at mobile instants of time, i.e., in (1) the set $E = \{\tau_1, \tau_2, \ldots\}$ is dependent of the different solutions. On the other hand, we point out that both proposition 3.1 and proposition 3.2 only show the persistence of stability of system (1) under certain impulsive perturbations because the corresponding system without impulse perturbations admit the same stability properties. This observation is clear from condition (4) or (7) and Lyapunov stability theory for differential systems without impulses.

3.2. New Stability Results

Theorem 3.1. *Assume that for system (1)* $f(0,t) = 0$ *and* $I_k(0) = 0$ *for all* $t \in R^+$ *and* $k \in \boldsymbol{N}$*, that there exists an* $h \in C(R^+, R^+)$*, nondecreasing, a* $\psi \in \Omega_1$*, a* $V : S(\rho) \times R^+ \to R^+$*, and* $\phi_1, \phi_2 \in \boldsymbol{K}$ *such that (2) holds for all* $(x, t) \in$

$S(\rho) \times R^+$, where $S(\rho) = \{x \in R^n : ||x|| < \rho\}$, $\Omega_1 = \{\psi \in C(R^+, R^+)$: strictly increasing, $\psi(0) = 0$, $\psi(s) < s$ for $s > 0\}$.

1) Assume that for any solution $x(t)$ of (1) which is defined on $[t_0, \infty)$, $V(x(t), t)$ is left continuous on $[t_0, \infty)$, and is differentiable everywhere on (t_0, ∞) except on the set $E = \{\tau_1, \tau_2, \ldots\}$ where E is the set of times when jumps occur for $x(t)$, and that there exists $\rho_0 > 0$ such that $x \in S(\rho_0)$ implies that $x + I_n(x) \in S(\rho)$ and

$$V(x(\tau_n^+), \tau_n^+) \leq \psi(V(x(\tau_n), \tau_n)) \tag{8}$$

for $n = 0, 1, \ldots$, where $\tau_0 = t_0$, $I_0 = 0$. Furthermore, assume that (5) holds for all $t \in (\tau_n, \tau_{n+1}]$ and $n = 0, 1, \ldots$, and that there exists $\gamma > 0$ such that

$$\psi^{-1}(a) > h(a) \quad \text{for} \quad \forall a \in (0, \gamma), \tag{9}$$

where ψ^{-1} is the inverse of the function ψ. Then the equilibrium $x = 0$ of system (1) is *uniformly stable*.

2) If in addition to the conditions in 1) (without conditions (5) and (9)) we assume that there exist a $\lambda \in C(R^+, R^+)$, a $H \in C(R^+, R^+)$ such that $H \circ \phi_1^{-1} \in \Omega_2$ and

$$\frac{dV(x(t), t)}{dt} \leq \lambda(t) H(||x(t)||), \quad t \neq \tau_n \tag{10}$$

is true for all $n \in \mathbf{N}$, where $\Omega_2 = \{\varphi \in C(R^+, R^+) : \varphi(0) = 0, \varphi(s) > 0$ for $s > 0$ and φ is nondecreasing$\}$. Furthermore, assume that there exist constants $\beta \geq \alpha > 0$ and $A > 0$ such that $\alpha \leq \tau_n - \tau_{n-1} \leq \beta$ and

$$\int_{\psi(\mu)}^{\mu} \frac{du}{H \circ \phi_1^{-1}(u)} - \int_{\tau_{n-1}}^{\tau_n} \lambda(s)\, ds \geq A \tag{11}$$

are true for all $n \in \mathbf{N}$ and $\mu \in (0, \infty)$. Then the equilibrium $x = 0$ of system (1) is *uniformly asymptotically stable*.

Proof:

1) We will prove that the equilibrium $x = 0$ is uniformly stable by definition, i.e., we will show that for any $\varepsilon \in (0, \rho_0]$, we can always find a $\delta = \delta(\varepsilon) > 0$ such that for all t_0, $||x(t)|| < \varepsilon$ for $t \geq t_0$ whenever $||x(t_0)|| < \delta$. To this end, we first let $\delta = \delta(\varepsilon) > 0$ be such that $\phi_2(\delta) < \gamma$ and $\psi^{-1}(\phi_2(\delta)) < \phi_1(\varepsilon)$.

For any $t_0(= \tau_0)$, the condition $||x(t_0)|| < \delta$ implies that

$$V(x(t_0), t_0) \leq \phi_2(||x(t_0)||) < \phi_2(\delta).$$

We claim that

$$V(x(t), t) \leq \psi^{-1}(\phi_2(\delta)), \quad \tau_0 < t \leq \tau_1. \tag{12}$$

Otherwise, there must exist a $\bar{t} \in (\tau_0, \tau_1]$ such that

$$V(x(\bar{t}), \bar{t}) > \psi^{-1}(\phi_2(\delta)) > \phi_2(\delta) > V(x(\tau_0), \tau_0). \tag{13}$$

In view of the continuity of $V(x(t), t)$ on $[\tau_0, \tau_1]$, we see that there exists $\hat{t} \in (\tau_0, \bar{t})$ such that

$$V(x(\hat{t}), \hat{t}) = \psi^{-1}(\phi_2(\delta)). \tag{14}$$

Thus, we have

$$\psi^{-1}(\phi_2(\delta)) = V(x(\hat{t}), \hat{t}) \leq h(V(x(\tau_0), \tau_0)) \leq h(\phi_2(\delta)).$$

This contradicts (9) and so (12) holds. From (8) and (12) we have

$$V(x(\tau_1^+), \tau_1^+) \leq \psi(V(x(\tau_1), \tau_1)) \leq \psi(\psi^{-1}(\phi_2(\delta))) = \phi_2(\delta).$$

Similarly, we can prove that

$$V(x(t), t) \leq \psi^{-1}(\phi_2(\delta)), \quad \tau_1 < t \leq \tau_2,$$

$$V(x(\tau_2^+), \tau_2^+) \leq \phi_2(\delta).$$

By the induction, it is not difficult to prove that

$$V(x(t), t) \leq \psi^{-1}(\phi_2(\delta)), \quad \tau_n < t \leq \tau_{n+1}, \quad n = 0, 1, \ldots.$$

Therefore, we can conclude that for all $t \geq t_0$

$$||x(t)|| \leq \phi_1^{-1}(V(x(t), t)) \leq \phi_1^{-1}(\psi^{-1}(\phi_2(\delta)))) < \phi_1^{-1}(\phi_1(\varepsilon)) = \varepsilon.$$

We have proved that the equilibrium $x = 0$ of system (1) is uniformly stable.

2) We will first prove that the equilibrium $x = 0$ is uniformly stable by definition. Observe that we now must use the conditions in 2) (without conditions (5) and (9)). However, along the proof in 1), we still must prove that (12) holds. If (12) does not hold, then we have again that (13) and (14) hold. By (13) there exists a $\check{t} \in (\tau_0, \hat{t})$ such that

$$V(x(\check{t}), \check{t}) = \phi_2(\delta). \tag{15}$$

Now (10) implies that for $t \neq \tau_n$

$$\frac{dV(x(t), t)}{dt} \leq \lambda(t)H(\phi_1^{-1}(V(x(t), t))) = \lambda(t)G(V(x(t), t)), \tag{16}$$

where $G = H \circ \phi_1^{-1}$. Integrating (16) over $(\check{t}, \hat{t})$ yields

$$\int_{V(x(\check{t}),\check{t})}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)} \leq \int_{\check{t}}^{\hat{t}} \lambda(s)\, ds \leq \int_{\tau_0}^{\tau_1} \lambda(s)\, ds. \tag{17}$$

On the other hand, let $\mu = \psi^{-1}(\phi_2(\delta))$ in (11). We have by (14) and (15)

$$\int_{V(x(\check{t}),\check{t})}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)} = \int_{\phi_2(\delta)}^{\psi^{-1}(\phi_2(\delta))} \frac{du}{G(u)}$$
$$\geq \int_{\tau_0}^{\tau_1} \lambda(s)\, ds + A > \int_{V(x(\check{t}),\check{t})}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)}.$$

This contradiction shows that (12) is true and so

$$V(x(\tau_1^+), \tau_1^+) \leq \psi(V(x(\tau_1), \tau_1)) \leq \phi_2(\delta).$$

We now show that

$$V(x(t), t) \leq \psi^{-1}(\phi_2(\delta)), \quad \tau_1 < t \leq \tau_2. \tag{18}$$

If (18) does not holds, then there exists a $\bar{t} \in (\tau_1, \tau_2]$ such that

$$V(x(\bar{t}), \bar{t}) > \psi^{-1}(\phi_2(\delta)) > \phi_2(\delta) \geq V(x(\tau_1^+), \tau_1^+).$$

By the continuity of $V(x(t), t)$ on $(\tau_1, \tau_2]$, we see that there exists $\hat{t} \in (\tau_1, \bar{t}]$ such that

$$V(x(\hat{t}), \hat{t}) = \psi^{-1}(\phi_2(\delta)).$$

Also, there exists a $\check{t} \in [\tau_1, \hat{t})$ such that

$$V(x(\check{t}^+), \check{t}^+) = \phi_2(\delta),$$

where $V(x(\check{t}^+), \check{t}^+) = V(x(\check{t}), \check{t})$ whenever $\check{t} \neq \tau_1$. Now, from (16) we have

$$\int_{V(x(\check{t}^+),\check{t}^+)}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)} \leq \int_{\check{t}}^{\hat{t}} \lambda(s)\, ds \leq \int_{\tau_1}^{\tau_2} \lambda(s)\, ds.$$

On the other hand, let $\mu = \psi^{-1}(\phi_2(\delta))$ in (11) we have

$$\int_{V(x(\check{t}^+),\check{t}^+)}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)} = \int_{\phi_2(\delta)}^{\psi^{-1}(\phi_2(\delta))} \frac{du}{G(u)}$$
$$\geq \int_{\tau_1}^{\tau_2} \lambda(s)\, ds + A > \int_{V(x(\check{t}^+),\check{t}^+)}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)}.$$

□

This is a contradiction and so (18) is true. Now by similar arguments to the proof in 1) we can conclude that the equilibrium $x = 0$ of system (1) is uniformly stable.

For $\varepsilon = \rho_0$, we can choose a $\delta = \delta(\rho_0) > 0$ such that $\psi^{-1}(\phi_2(\delta)) \leq \phi_1(\rho_0)$, and in view of the proof of uniform stability, we know that the condition $||x(t_0)|| <$

δ implies that

$$V(x(t), t) \leq \psi^{-1}(\phi_2(\delta)) \text{ and } ||x(t)|| < \rho_0, \quad t \geq t_0.$$

Now, let any $\varepsilon > 0 (\varepsilon < \rho_0)$ be given. We will find a $T = T(\varepsilon) > 0$ such that

$$||x(t)|| < \varepsilon, \quad t \geq t_0 + T, \tag{19}$$

which yields the uniform asymptotic stability of the equilibrium $x = 0$ of system (1) by definition. To this end, we take the smallest positive integer $N = N(\varepsilon)$ such that

$$\psi^{-1}(\phi_2(\delta)) \leq \psi(\phi_1(\varepsilon)) + NAG(\psi(\phi_1(\varepsilon))). \tag{20}$$

Let $\chi_i = [\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots$, where $\tau_0 = t_0$. Since

$$V(x(\tau_k^+), \tau_k^+) \leq \psi(V(x(\tau_k), \tau_k)) \leq V(x(\tau_k), \tau_k), \quad k \in \mathbf{N},$$

it follows that $\sup\{V(x(t), t) : t \in \chi_i\} = V(x(r_i), r_i)$ for some $r_i \in \chi_i$. Set $T = N\beta$. We will prove that (19) holds for this T. To this end, we first prove that if

$$V(x(r_i), r_i) \leq \psi(\phi_1(\varepsilon)) \quad \text{for some} \quad i \in \{1, 2, \ldots, N\}, \tag{21}$$

then

$$V(x(t), t) \leq \phi_1(\varepsilon), \quad t \geq \tau_N. \tag{22}$$

Indeed, from (21) we have

$$V(x(t), t) \leq \psi(\phi_1(\varepsilon)) < \phi_1(\varepsilon), \quad \tau_{i-1} \leq t \leq \tau_i. \tag{23}$$

We claim that

$$V(x(t), t) \leq \phi_1(\varepsilon), \quad \tau_i < t \leq \tau_{i+1}. \tag{24}$$

Otherwise, there must be a $\bar{r} \in (\tau_i, \tau_{i+1}]$ such that

$$V(x(\bar{r}), \bar{r}) > \phi_1(\varepsilon) > \psi(\phi_1(\varepsilon)) \geq V(x(\tau_i), \tau_i) \geq V(x(\tau_i^+), \tau_i^+).$$

Thus, there exist $\hat{r} \in (\tau_i, \bar{r})$ and $\check{r} \in [\tau_i, \hat{r})$ such that

$$V(x(\hat{r}), \hat{r}) = \phi_1(\varepsilon), \quad V(x(\check{r}^+), \check{r}^+) = \psi(\phi_1(\varepsilon)).$$

Integrating (16) from $\check{r}$ to $\hat{r}$ yields

$$\begin{aligned}\int_{V(x(\check{r}^+),\check{r}^+)}^{V(x(\hat{r}),\hat{r})} \frac{du}{G(u)} &\leq \int_{\check{r}}^{\hat{r}} \lambda(s)\,ds < \int_{\tau_i}^{\tau_{i+1}} \lambda(s)\,ds + A \\ &\leq \int_{\psi(\phi_1(\varepsilon))}^{\phi_1(\varepsilon)} \frac{du}{G(u)} = \int_{V(x(\check{r}^+),\check{r}^+)}^{V(x(\hat{r}),\hat{r})} \frac{du}{G(u)}.\end{aligned}$$

This is a contradiction and so (24) holds. From (24) and (8) we have

$$V(x(\tau_{i+1}^+), \tau_{i+1}^+) \leq \psi(V(x(\tau_{i+1}), \tau_{i+1})) \leq \psi(\phi_1(\varepsilon)) < \phi_1(\varepsilon).$$

Similarly, we can prove that

$$V(x(t),t) \le \phi_1(\varepsilon), \quad \tau_{i+1} < t \le \tau_{i+2}, \quad V(x(\tau_{i+2}^+),\tau_{i+2}^+) \le \psi(\phi_1(\varepsilon)).$$

By the induction, it is easy to prove that (22) holds.

Next, we will show that (21) holds for some $i \in \{1, 2, \dots, N\}$. Assume that $V(x(r_i), r_i) > \psi(\phi_1(\varepsilon))$ for all $i = 1, 2, \dots, N$. In the following proof, we will derive a contradiction. To this end, we prove that

$$V(x(r_i), r_i) \le V(x(r_0), r_0) - iAG(\psi(\phi_1(\varepsilon))), i = 0, 1, \dots, N, \tag{25$_i$}$$

where $V(x(r_0), r_0) = \psi^{-1}(\phi_2(\delta))$. Clearly, $(25)_i$ holds for $i = 0$. We now assume that $(25)_i$ holds for some $i(0 < i < N)$. We will prove that $(25)_{i+1}$ holds. We first prove that

$$V(x(r_{i+1}), r_{i+1}) \le V(x(r_i), r_i). \tag{26}$$

Indeed, since $V(x(t), t) \le V(x(r_i), r_i)$ for $\tau_{i-1} \le t \le \tau_i$, it follows that

$$V(x(\tau_i^+), \tau_i^+) \le \psi(V(x(\tau_i), \tau_i)) \le \psi(V(x(r_i), r_i)).$$

We claim that

$$V(x(t), t) \le V(x(r_i), r_i), \quad \tau_i < t \le \tau_{i+1}. \tag{27}$$

If (27) does not hold, then there exists a $\bar{t} \in (\tau_i, \tau_{i+1}]$ such that

$$V(x(\bar{t}), \bar{t}) > V(x(r_i), r_i) > \psi(V(x(r_i), r_i)) \ge \psi(V(x(\tau_i), \tau_i)) \ge V(x(\tau_i^+), \tau_i^+).$$

Thus, there exist $\hat{t} \in (\tau_i, \bar{t})$ and $\check{t} \in [\tau_i, \hat{t})$ such that

$$V(x(\hat{t}), \hat{t}) = V(x(r_i), r_i), \quad V(x(\check{t}^+), \check{t}^+) = \psi(V(x(r_i), r_i)). \tag{28}$$

Integrating (16) from $\check{t}$ to $\hat{t}$ and using (28) we have

$$\begin{aligned}\int_{V(x(\check{t}^+),\check{t}^+)}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)} &\le \int_{\check{t}}^{\hat{t}} \lambda(s)\,ds < \int_{\tau_i}^{\tau_{i+1}} \lambda(s)\,ds + A \\ &\le \int_{\psi(V(x(r_i),r_i))}^{V(x(r_i),r_i)} \frac{du}{G(u)} = \int_{V(x(\check{t}^+),\check{t}^+)}^{V(x(\hat{t}),\hat{t})} \frac{du}{G(u)}.\end{aligned}$$

This is a contradiction and so (27) holds. By (27) and the definition of $V(x(r_{i+1}), r_{i+1})$, we see that (26) is true. We consider two possible cases.

Case 1. $\psi(\phi_1(\varepsilon)) < V(x(r_{i+1}), r_{i+1}) \le \psi(V(x(r_i), r_i))$.

In this case, from

$$\int_{V(x(r_{i+1}),r_{i+1})}^{\psi^{-1}(V(x(r_{i+1}),r_{i+1}))} \frac{du}{G(u)} \ge A,$$

we have

$$\begin{aligned} V(x(r_{i+1}), r_{i+1}) &\le \psi^{-1}(V(x(r_{i+1}), r_{i+1})) - AG(\psi(\phi_1(\varepsilon))) \\ &\le V(x(r_i), r_i) - AG(\psi(\phi_1(\varepsilon))) \\ &\le V(x(r_0,)r_0) - (i+1)AG(\psi(\phi_1(\varepsilon))), \end{aligned}$$

which implies $(25)_{i+1}$.

Case 2. $\psi(V(x(r_i), r_i)) < V(x(r_{i+1}), r_{i+1}) \le V(x(r_i), r_i)$.

In this case, if $r_{i+1} \in \chi_{i+1} - \{\tau_i\}$, from

$$V(x(\tau_i^+), \tau_i^+) \le \psi(V(x(\tau_i), \tau_i)) \le \psi(V(x(r_i), r_i)),$$

we see that there is a $\bar{r} \in [\tau_i, r_{i+1})$ such that

$$V(x(\bar{r}^+), \bar{r}^+) = \psi(V(x(r_i), r_i)). \tag{29}$$

Integrating (16) from $\bar{r}$ to r_{i+1} yields

$$\int_{V(x(\bar{r}^+),\bar{r}^+)}^{V(x(r_{i+1}),r_{i+1})} \frac{du}{G(u)} \le \int_{\bar{r}}^{r_{i+1}} \lambda(s)\,ds \le \int_{\tau_i}^{\tau_{i+1}} \lambda(s)\,ds,$$

which, together with (29) and condition (8), implies that

$$\int_{\psi(V(x(r_i),r_i))}^{V(x(r_{i+1}),r_{i+1})} \frac{du}{G(u)} \le -A + \int_{\psi(V(x(r_i),r_i))}^{V(x(r_i),r_i)} \frac{du}{G(u)}.$$

According,

$$\int_{V(x(r_{i+1}),r_{i+1})}^{V(x(r_i),r_i)} \frac{du}{G(u)} \ge A. \tag{30}$$

It follows that

$$\begin{aligned} V(x(r_{i+1}), r_{i+1}) &\le V(x(r_i), r_i) - AG(\psi(\phi_1(\varepsilon))) \\ &\le V(x(r_0), r_0) - (i+1)AG(\psi(\phi_1(\varepsilon))), \end{aligned}$$

which implies $(25)_{i+1}$. If $r_{i+1} = \tau_i$, then from

$$V(x(\tau_{i-1}^+), \tau_{i-1}^+) \le \psi(V(x(\tau_{i-1}), \tau_{i-1})) \le \psi(V(x(r_i), r_i)),$$

we have

$$\begin{aligned} \int_{\psi(V(x(r_i),r_i))}^{V(x(r_{i+1}),r_{i+1})} \frac{du}{G(u)} &\le \int_{V(x(\tau_{i-1}^+),\tau_{i-1}^+)}^{V(x(\tau_i),\tau_i)} \frac{du}{G(u)} \le \int_{\tau_{i-1}}^{\tau_i} \lambda(s)\,ds \\ &\le -A + \int_{\psi(V(x(r_i),r_i))}^{V(x(r_i),r_i)} \frac{du}{G(u)}. \end{aligned}$$

We have again that (30) holds. By combining the cases 1 and 2, we may conclude that $(25)_{i+1}$ holds. By the induction, we see that $(25)_i$ hold for all $i = 0, 1, \dots, N$.

Therefore, by (20), we have

$$\begin{aligned}V(x(r_N), r_N) &\le V(x(r_0), r_0) - NAG(\psi(\phi_1(\varepsilon)))\\ &= \psi^{-1}(\phi_2(\delta)) - NAG(\psi(\phi_1(\varepsilon)))\\ &\le \psi(\phi_1(\varepsilon)).\end{aligned}$$

This contradicts the assumption that $V(x(r_i), r_i) > \psi(\phi_1(\varepsilon))$ for all $i = 1, 2, \ldots, N$. Thus, we have proved that there exists a $i \in \{1, 2, \ldots, N\}$ such that (21) holds and so (22) holds as in the preceding proof. Since $t_0 + T = t_0 + N\beta \ge \tau_N$, it follows that

$$||x(t)|| \le \phi_1^{-1}(V(x(t), t)) \le \phi_1^{-1}(\phi_1(\varepsilon)) = \varepsilon$$

is true for all $t \ge t_0 + T$. This proves that $x = 0$ is uniformly asymptotically stable.

Remarks.

1) The conditions (3) and (4) in Proposition 3.1 imply that the Lyapunov function V is required to be monotonically nonincreasing on $[t_0, \infty)$. Thus, Proposition 3.1 is a simple extendence of Lyapunov stability theorems to the system with impulse effects (1). In 1) of Proposition 3.2, the above monotonity is relaxed but $V(x(\tau_n^+), \tau_n^+)$ is required to be monotonically nonincreasing for $n = 0, 1, \ldots$. Specifically, the function h in condition (5) is required to satisfy $h(0) = 0$. Clearly, in Theorem 3.1, the conditions for uniform stability are partially different from those given in Propositions 3.1 and 3.2.

2) It should be emphasized that in Proposition 3.1 (resp. Proposition 3.2) the condition (4) (resp. (6)) for uniform asymptotic stability cannot yield new stability properties which are caused by impulsive perturbations. In the following, by using the new stability results in Theorem 3.1, we will give the application to show that certain impulsive perturbations may make a unstable system uniformly stable even uniformly asymptotically stable.

4. AN APPLICATION: IMPULSIVE CONTROL FOR STABILITY

Let a plant be a nonlinear system of the form

$$\frac{dx}{dt} = Ax + f(x, t), \quad y = Bx, \tag{31}$$

where $x \in R^n$ is state variable, A is an $n \times n$ constant matrix, $y \in R^m$ is output, and B is an $m \times n$ constant matrix. $f \in C(R^n \times R, R^n)$ is a nonlinear function. The control instant is given by $\tau_k, k = 1, 2, \ldots$. Then the nonlinear impulsive control system is given by

$$\begin{cases} \frac{dx}{dt} = Ax + f(x,t) \\ y = Bx, t \neq \tau_k, \\ \triangle x = Cy, t = \tau_k, \quad k = 1, 2, \ldots, \end{cases} \tag{32}$$

where C is an $n \times m$ constant matrix.

Remark. The dynamics of the impulsive control system in (32) is governed by the ODE in (31) when $t \neq \tau_k$. It is clear that the controlled system is a free system whenever $t \neq \tau_k$. In this sense, impulsive control is totally different from continuous control where plants are continuously driven by inputs. Only at instant $\tau_k, k = 1, 2, \ldots$, the state variable is changed from $x(\tau_k^-)$ to $x(\tau_k^+) = x(\tau_k^-) + \triangle x|_{t=\tau_k} = x(\tau_k^-) + Cy$ instantaneously. Then the impulsive control system can be rewritten as

$$\begin{cases} \frac{dx}{dt} = Ax + f(x,t), \quad t \neq \tau_k, \\ \triangle x = CBx, \quad t = \tau_k, \quad k = 1, 2, \ldots, \\ x(t_0^+) = x_0. \end{cases} \tag{33}$$

Theorem 4.1. *Let the $n \times n$ matrix D be symmetric and positive definite, and $\lambda_i > 0 (i = 1, 2)$ be the smallest and the largest eigenvalues of D, respectively. Let $P = DA + A^T D$, where A^T is the transpose of A. Let λ_3 be the largest eigenvalue of $D^{-1}P$, and λ_4 be the largest eigenvalue of $D^{-1}[I + (CB)^T]D(I + CB)$ such that $\lambda_4 \in (0, 1)$, where I is the identity matrix. Assume that $||f(x,t)|| \leq L(t)||x||$ for $(x,t) \in S(\rho) \times [t_0, \infty)$, where $S(\rho) = \{x \in R^n : ||x|| < \rho\}$, and $L \in C([t_0, \infty), R^+)$. Then the equilibrium $x = 0$ of system (33) is uniformly asymptotically stable if there exist constants $0 < \alpha \leq \beta$ and $\gamma > 1$ such that $\alpha \leq \tau_k - \tau_{k-1} \leq \beta$ and*

$$\int_{\tau_{k-1}}^{\tau_k} \lambda(s)\,ds + \ln(\gamma\lambda_4) \leq 0, k = 1, 2, \ldots, \tag{34}$$

where $\tau_0 = t_0$ and

$$\lambda(t) = \left(\lambda_3 + 2L(t)\sqrt{\frac{\lambda_2}{\lambda_1}}\right) \geq 0.$$

Proof: We construct a Lyapunov function $V(x,t) = V(x) = x^T Dx$. Then, for any solution $x = x(t)$ of (33), when $t \neq \tau_k$, we have

$$\begin{aligned} dV(x)/dt &= x^T(A^T D + DA)x + (f^T(x,t)Dx + x^T Df(x,t)) \\ &= x^T Px + (f^T(x,t)Dx + x^T Df(x,t)) \end{aligned}$$

$$\leq \left(\lambda_3 + 2L(t)\sqrt{\frac{\lambda_2}{\lambda_1}}\right) V(x) = \lambda(t)V(x).$$

□

It is clear that

$$\lambda_1 ||x||^2 \leq V(x) \leq \lambda_2 ||x||^2.$$

Thus, one can see that conditions (2) and (10) of Theorem 3.1 are satisfied. Set $\rho_0 = \rho/||I + CB||$. Then for $x \in S(\rho_0)$, we have

$$||x + I_k(x)|| = ||x + CBx|| \leq ||I + CB|| ||x|| \leq \rho.$$

When $t = \tau_k$, we have

$$\begin{aligned} V(x + CBx) &= (x + CBx)^T D(x + CBx) \\ &= x^T (I + (CB)^T) D(I + CB)x \leq \lambda_4 V(x). \end{aligned}$$

Thus, condition (8) of Theorem 3.1 is satisfied for $\psi(s) = \lambda_4 s$. It is clear that condition (11) is satisfied for $A = \ln \gamma$. Therefore, by Theorem 3.1, the equilibrium $x = 0$ of (33) is uniformly asymptotically stable.

Remark. From Theorem 4.1, one can see that impulsive perturbations may make a unstable system uniformly stable even uniformly asymptotically stable. Indeed, consider the scalar differential system with impulse effects

$$\begin{cases} \frac{dx}{dt} = ax + x^3, & t \neq \tau_k, \\ \triangle x = bx, & t = \tau_k, k = 1, 2, \ldots, \\ x(t_0^+) = x_0, \end{cases}$$

where $x \in R, a \in (0, \infty), b \in R$ and $t_0 \leq \tau_1 < \tau_2 < \ldots$. By theorem 4.1, it is easy to see that the equilibrium $x = 0$ of the system is uniformly asymptotically stable if there exist constants $L > 0$ and $\gamma > 1$ such that

$$2(a + L)(\tau_k - \tau_{k-1}) + \ln(\gamma(1 + b)^2) \leq 0, \quad k = 1, 2, \ldots,$$

where $\tau_0 = t_0$. On the other hand, it is clear that the equilibrium $x = 0$ of the equation $(dx/dt) = ax + x^3$ is unstable.

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